Math 2114 : Lecture 12 : Column Space and Null Space

Definition: Suppose the $m \times n$ matrix A has column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$. We define the column space of A to be

$$\text{col}(A) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \tag{1}$$

Note 1: The column space is the set of *all* linear combinations of the column vectors of A.

Example 1: Consider the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 4 & -3 \end{bmatrix}$. Is the vector $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 11 \end{bmatrix}$ in col(A)? If so find a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 11 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 4 & -3 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 2 \\ 11 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 3 & -1 & 2 \\ 1 & 4 & -3 & 11 \end{array}\right] \underset{\sim}{\begin{array}{l} R_3 := R_3 - R_1 \\ R_2 := R_2 - 2R_1 \end{array}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 1 & -4 \\ 0 & 2 & -2 & 8 \end{array}\right] \underset{\sim}{R_3 := R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 - x_3 = 3$$
$$-x_2 + x_3 = -4$$
$$x_3 = 0, \quad x_2 = 4, \quad x_1 = -5$$

$$\vec{x} = \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix}$$

Yes!

Note 2: A vector $\mathbf{b}$ is in col(A) if and only if the linear system $A\mathbf{x} = \mathbf{b}$ is *consistent*.

Note 3: The linear system is $A\mathbf{x} = \mathbf{b}$ is consistent for all vectors $\mathbf{b}$ if and only if col$(A) = \mathbb{R}^m$. In this case we say that the column vectors of A *span* (or *generate*) $\mathbb{R}^m$.

Example 2: For example, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Add -1 times the first row to the second row to get $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Show that the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in col(A) but $\mathbf{b}$ is *not* in col(C).

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = C$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{b} \text{ is col}(A)$$

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lightning \qquad \vec{b} \text{ is not in col}(C)$$

Note 4: Elementary row operations usually *change* the column space of a matrix.